Regularized Linear Regression

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These slides are adopted based on slides from Andrew Zisserman, Jonathan Taylor, Chao Zhang, Mahdi Roozbahani and Yaser Abu-Mostafa.

Recap

- Linear regression:
- $Y = \theta X$ → Matrix
- MSE

$x_1 \quad \dots \quad x_n \quad 1$

$\theta_1 \quad \dots \quad \theta_n \quad \theta_{n+1}$ ← constant

$\hat{Y} = \hat{\theta} \cdot X$

$\hat{Y} \approx Y$

Polynomial regression

$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 \cdots$$

$$y = \theta_0 z_0 + \theta_1 z_1 + \cdots$$

$$z_n = x^n$$

Polynomial regression when order not known

$$g \propto x^{-2}$$

$x \rightarrow$ distance between objects

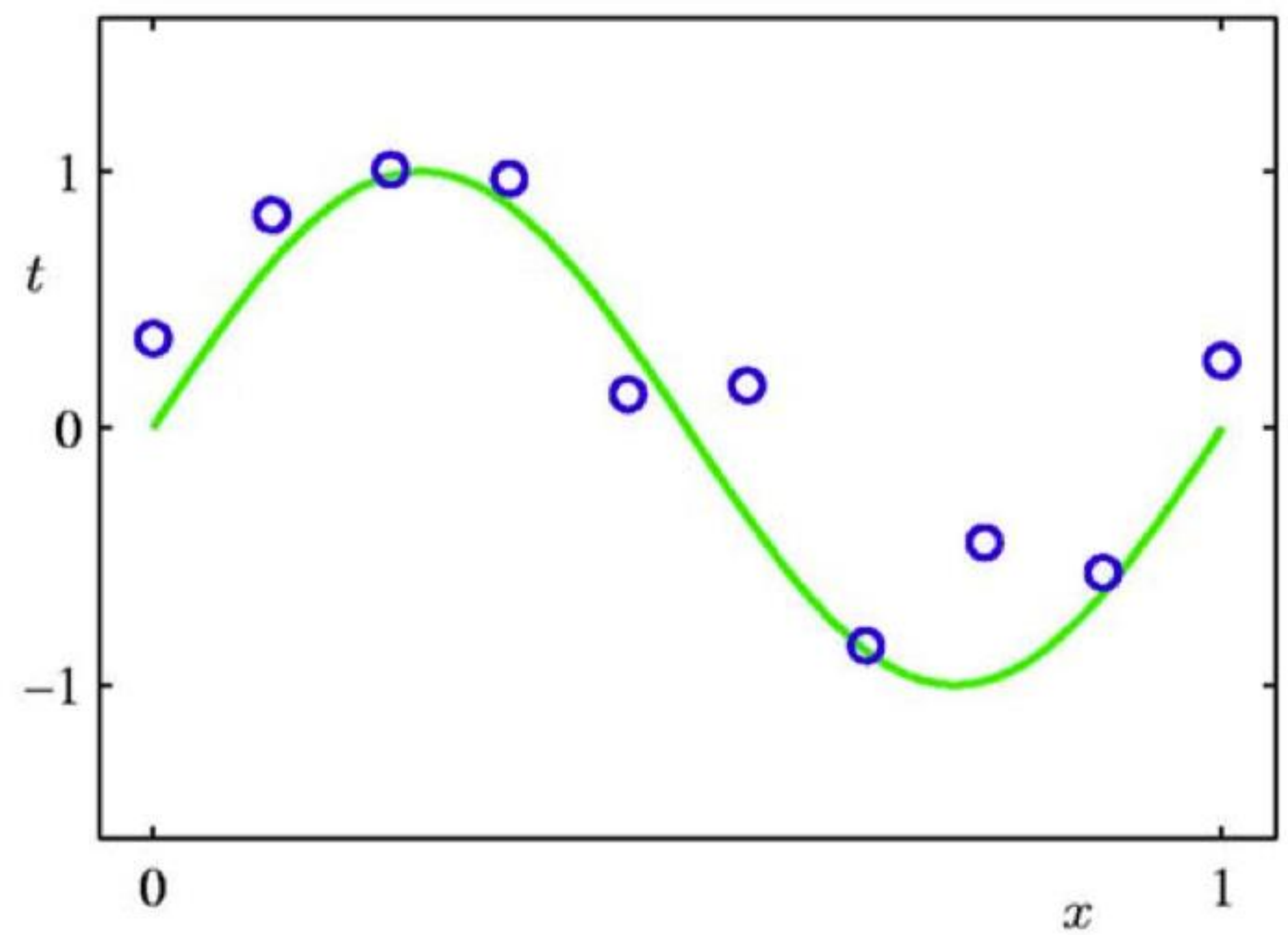
$g \rightarrow$ gravitational force b/w them

$$g = \theta_{-100}\, x^{-100} + \theta_{-99}\, x^{-99} \cdots\cdots + \theta_{100}\, x^{100}$$

Outline

- Overfitting and regularized learning
- Ridge regression
- Lasso regression
- Determining regularization strength

Regression: Recap

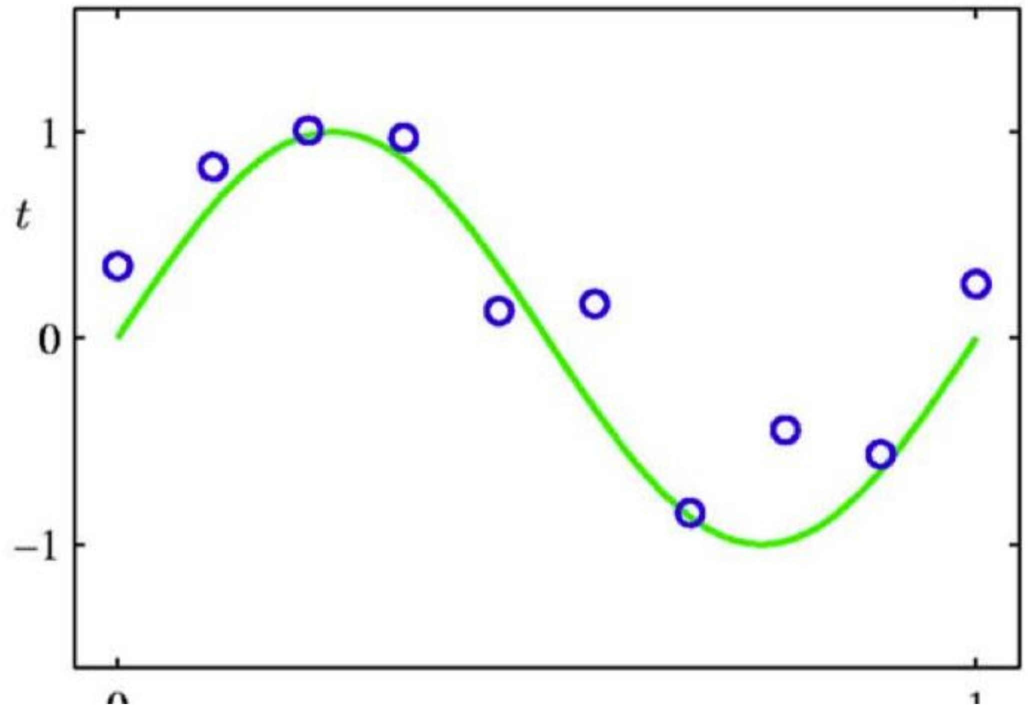


- Suppose we are given a training set of N observations

$$(x_1, \ldots, x_N) \text{ and } (y_1, \ldots, y_N)$$

- Regression problem is to estimate y(x) from this data

Regression: Recap



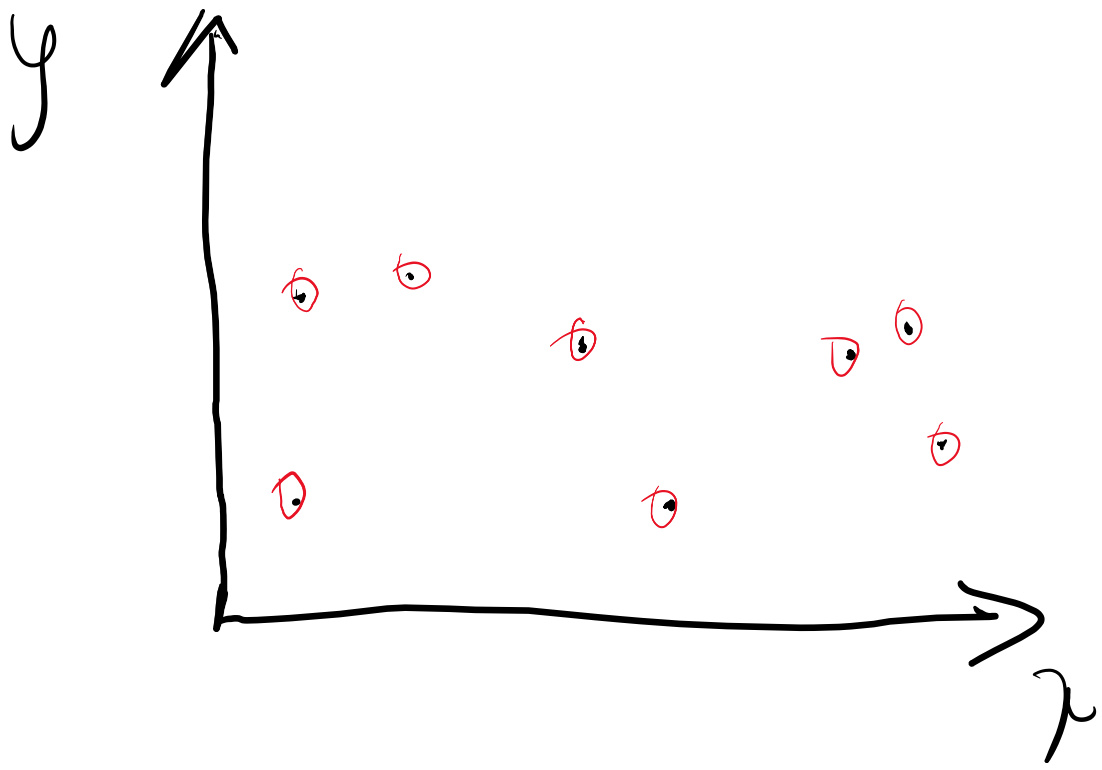
- Want to fit a polynomial regression model

$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_d x^d + \epsilon$$

- $z = \{1, x, x^2, \ldots, x^d\} \in R^d$ and $\theta = (\theta_0, \theta_1, \theta_2, \ldots, \theta_d)^T$

linear

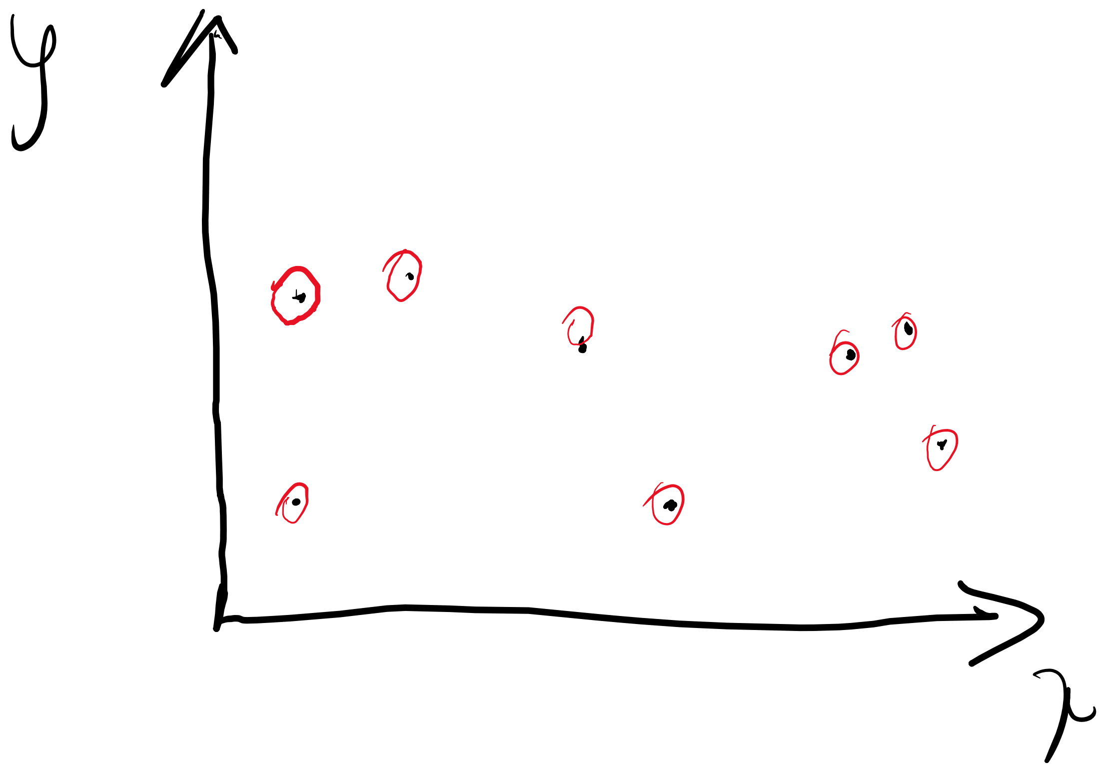
$$y = z\theta$$



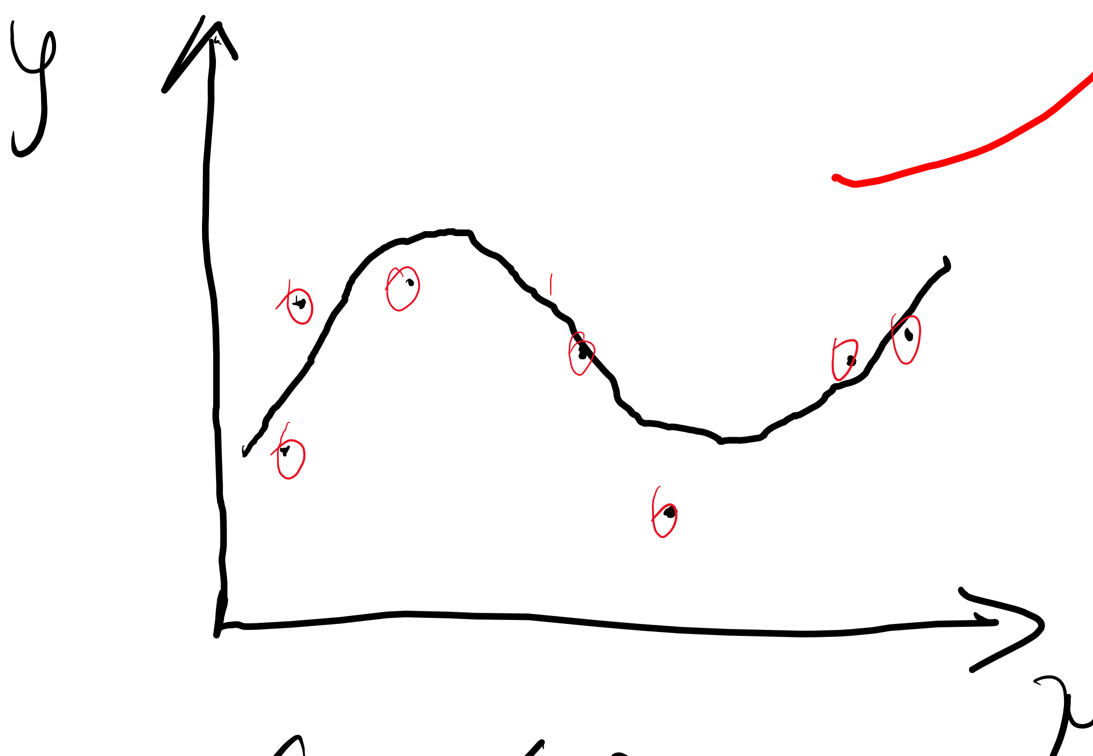
Real

Regression

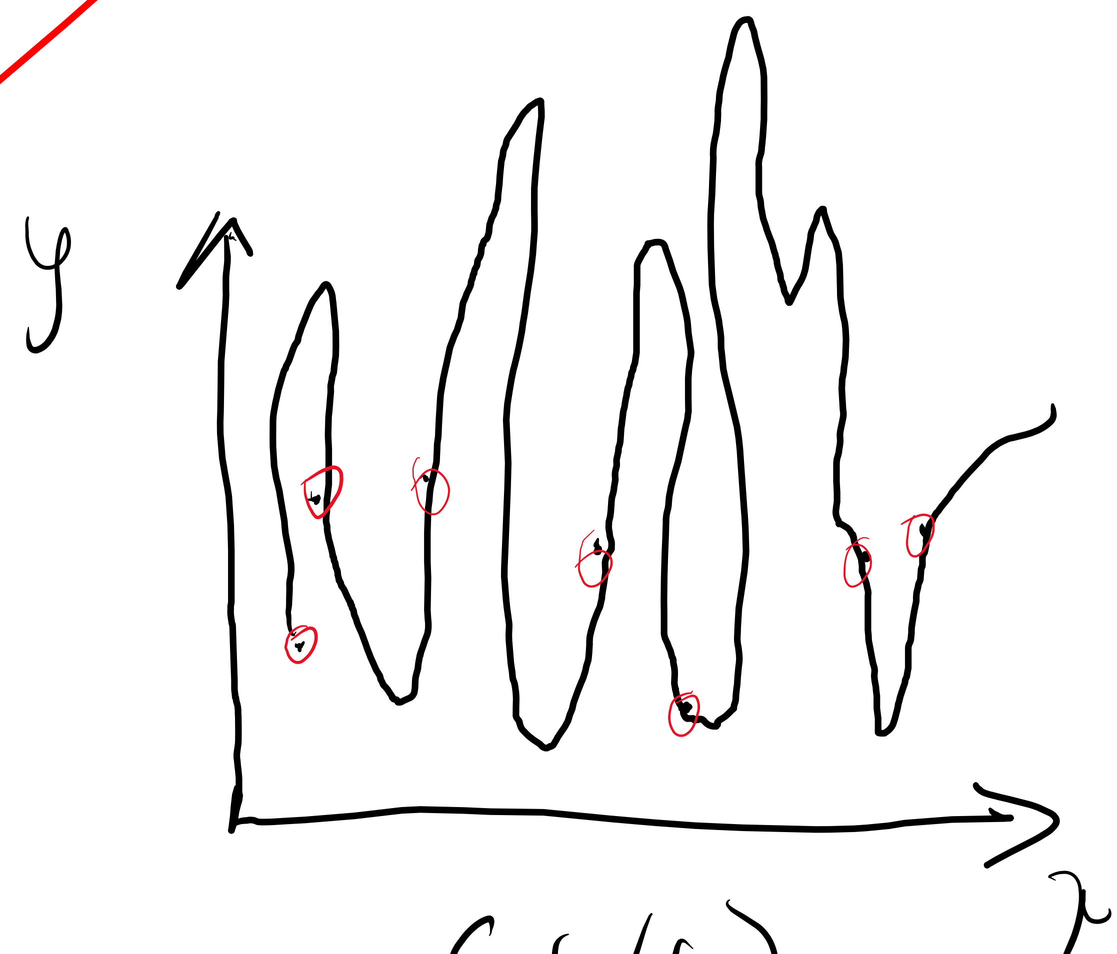
Problem!!!



generalization
because no overfitting

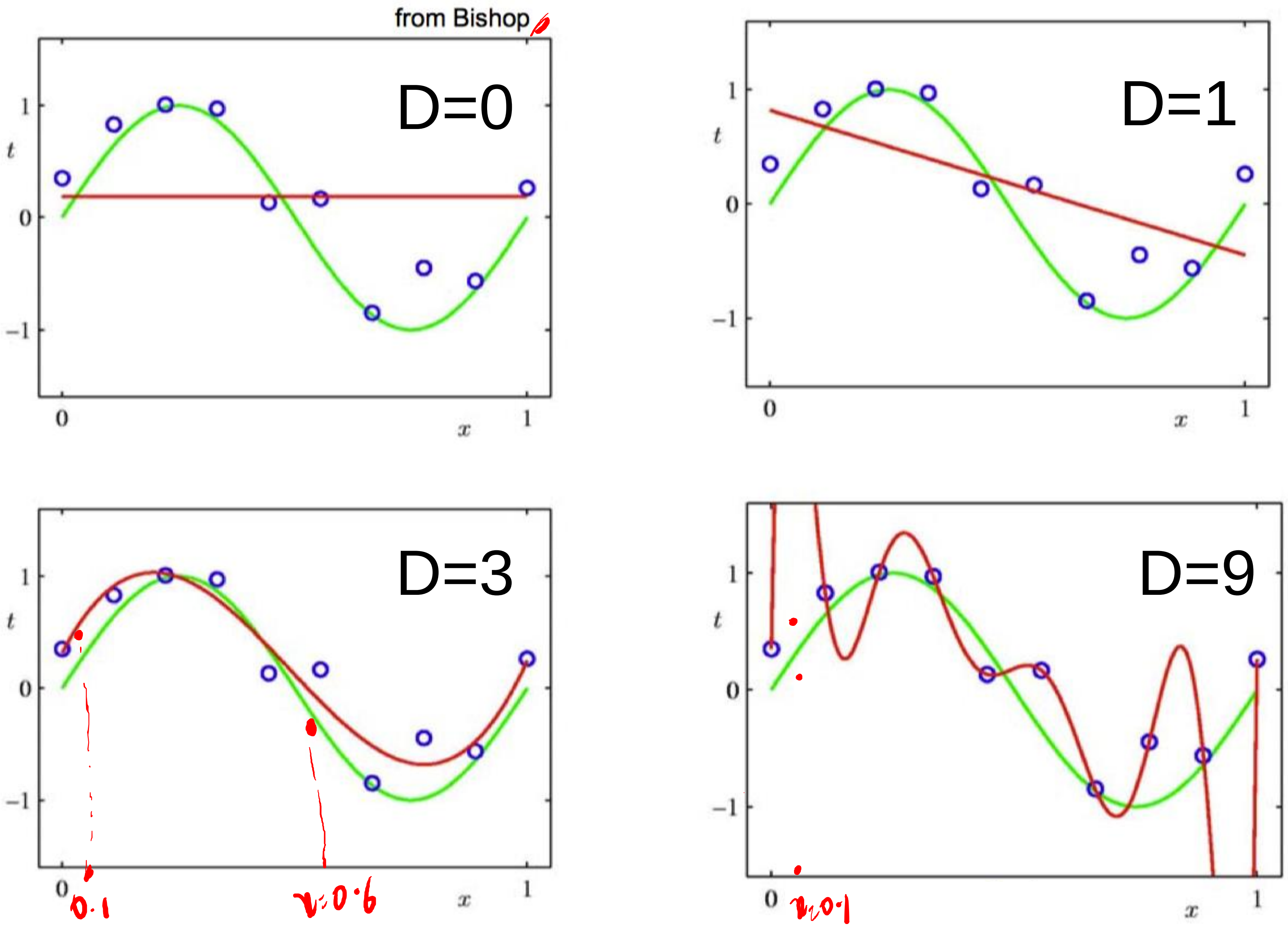


Sol. (a)



Sol. (b)

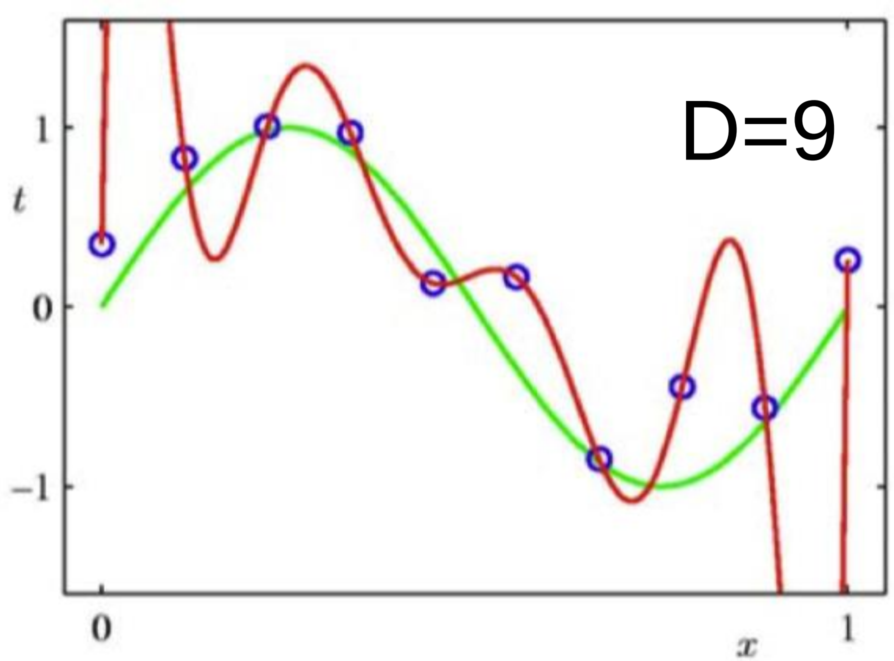
Which One is Better?

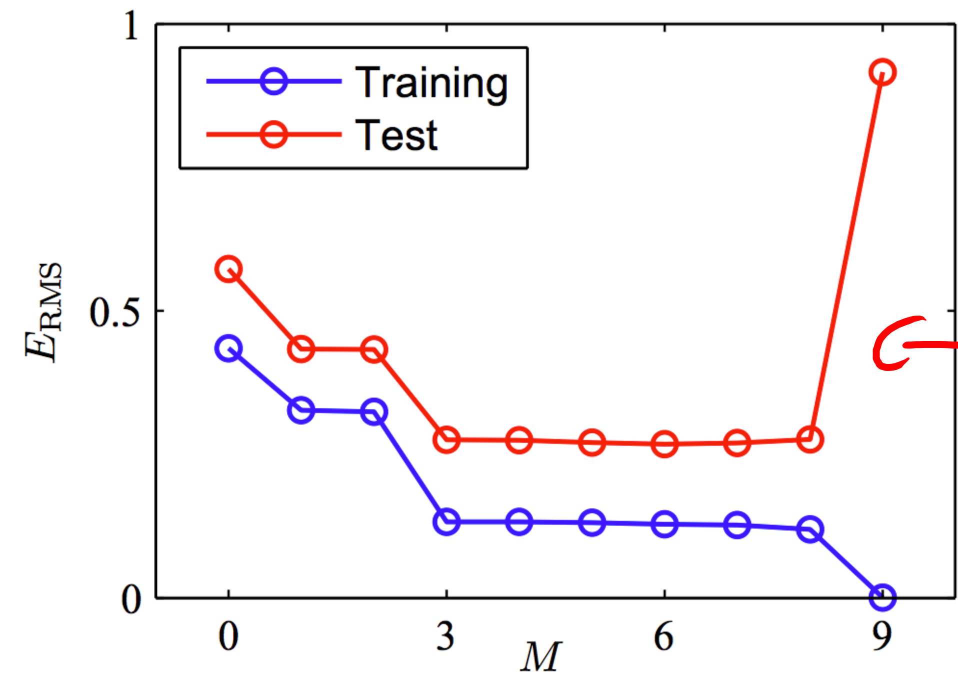


- Can we increase the maximal polynomial degree to very large, such that the curve passes through all training points?

No, this can lead to **overfitting**!

The Overfitting Problem

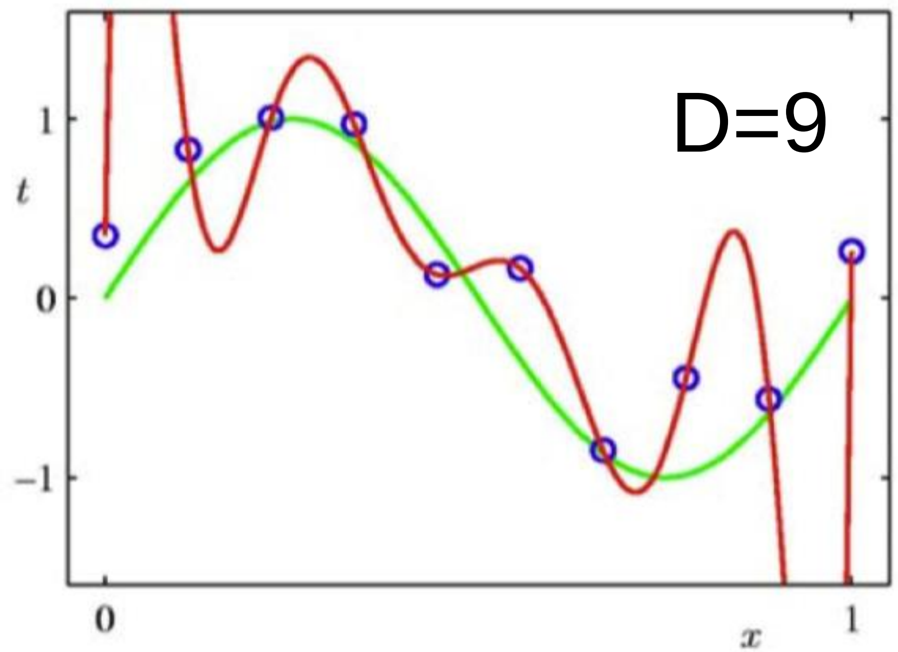




Symptom

- The training error is very low, but the error on test set is large.
- The model captures not only patterns but also noisy nuisances in the training data.

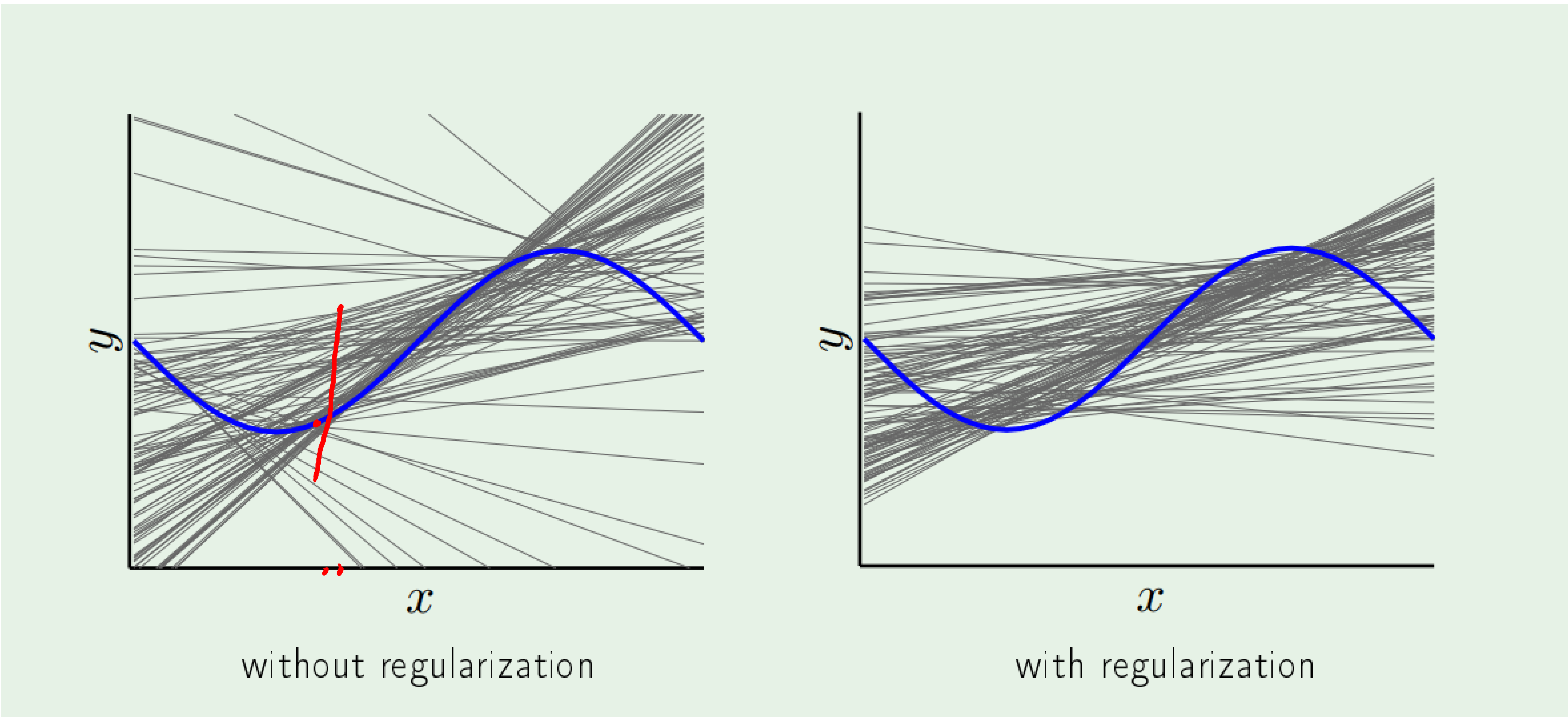
The Overfitting Problem



- In regression, overfitting is often associated with large Weights (severe oscillation)
- How can we address overfitting?

Regularization

(smart way to cure overfitting disease)



y(0.005) = 10

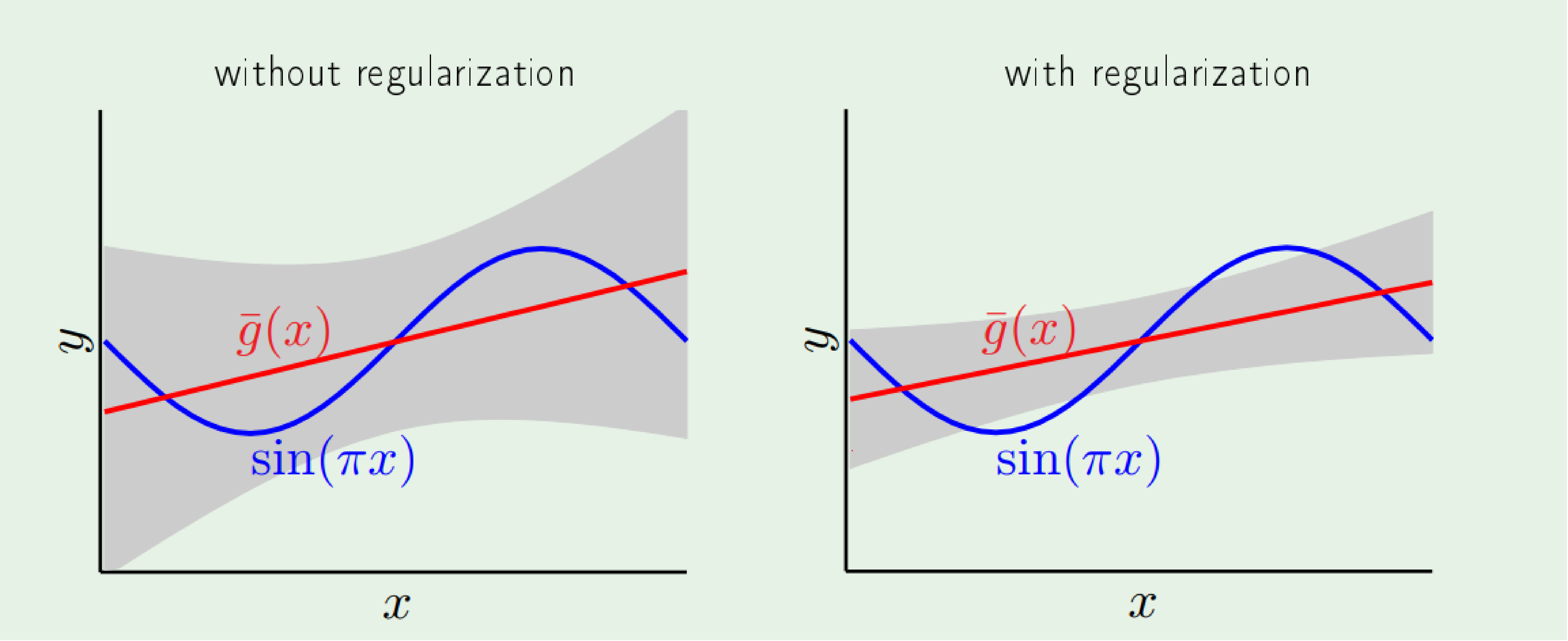
~~y(0.0051) = 10,000~~

Put a brake on fitting

Fit a linear line on sinusoidal with just two data points

Who is the winner?

$\bar{g}\,(x)$: average over all lines



bias=0.21; var=1.69

bias=0.23; var=0.33

Regularized Learning

Why this term leads to regularization of parameters

Minimize $$E(\theta) + \frac{\lambda}{N}\theta^T\theta$$

- Cost function – squared loss:

target value

$$\widetilde{E}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\underbrace{\{f(x_i,\theta) - y_i\}^2}_{\text{loss function}} + \underbrace{\frac{\lambda}{N}\|\theta\|^2}_{\text{regularization}}$$

Polynomial Model

Want to fit a polynomial regression model

$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_d x^d + \epsilon$$

Let's rewrite it as:

$$y = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + \cdots + \theta_d z_\mathrm{d} + \epsilon = \boldsymbol{z\theta}$$

$y = z\theta$

Regularizing is just constraining the weights ($\boldsymbol{\theta}$)

For example: let's do a **hard** constraining

$$y = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + \cdots + \theta_d z_\mathrm{d}$$

subject to

$$\theta_d = 0 \; for \; d > 2$$

$$y = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + 0 + \cdots + 0$$

Let's not penalize θ in such a harsh way

let's cut them some slack

$$\theta = argmin_{\theta} E(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left(y^i - z_i\theta\right)^2$$

$$Minimize\ \frac{1}{N}(z\theta - y)^T(z\theta - y)$$

Subject to $\theta^t\theta \leq C$

matrix form

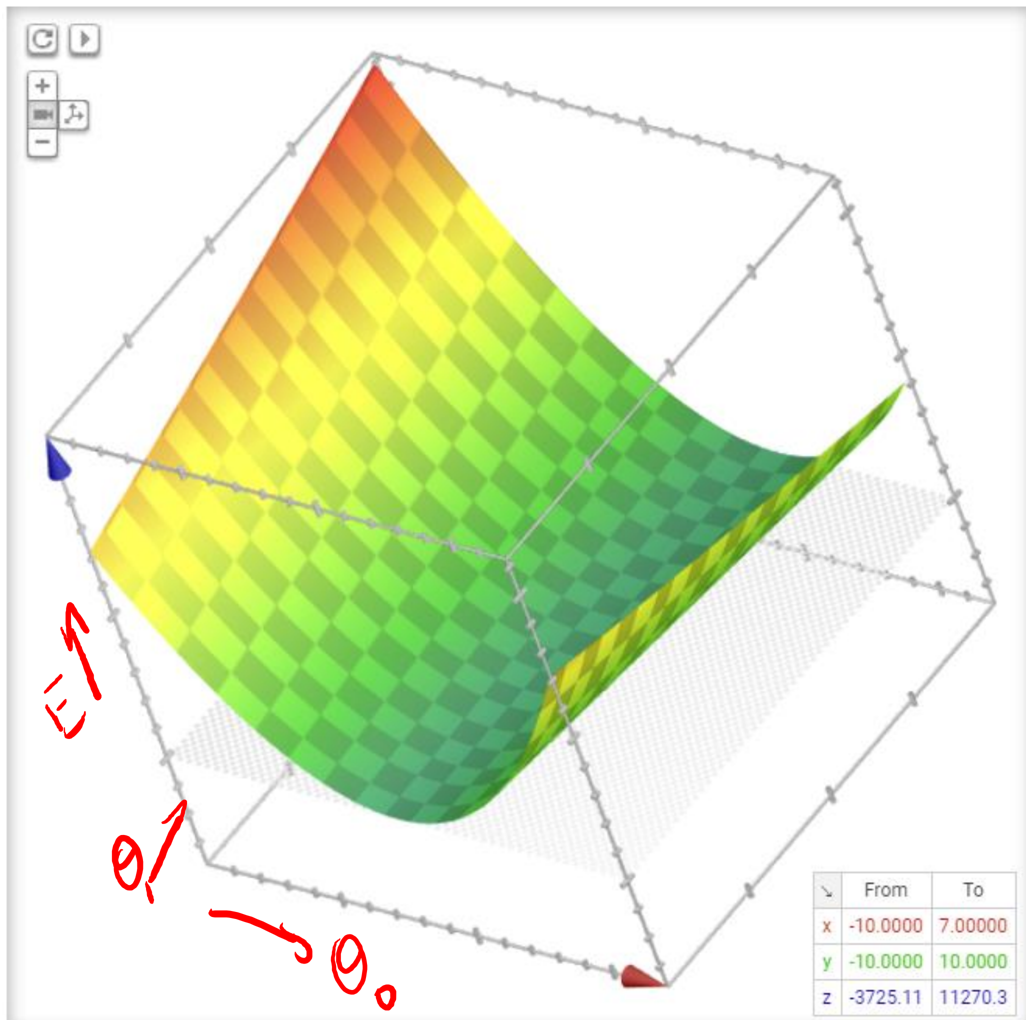
$\sum \theta_i^2 \leq c$

constant

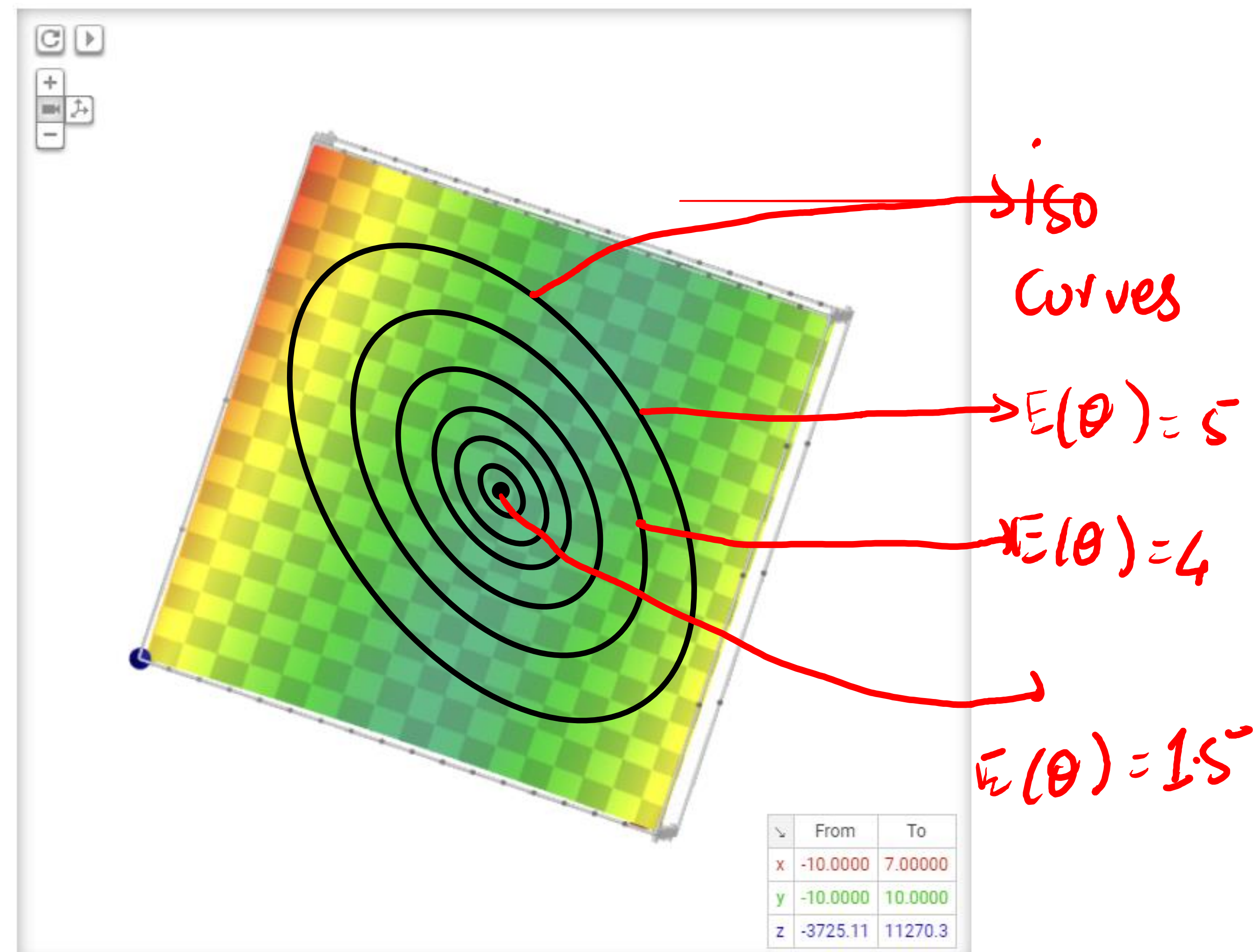
For simplicity let's call θ_{lin} as weights' solution for non constrained one and θ for the constrained model.

$$E(\theta) = \frac{1}{N}(z\theta - y)^T(z\theta - y)$$

Possible graph for $E(\theta)$ for different values of θ_0 and θ_1 and given observation data



3D view



Top view

Gradient of $\theta^t\theta$

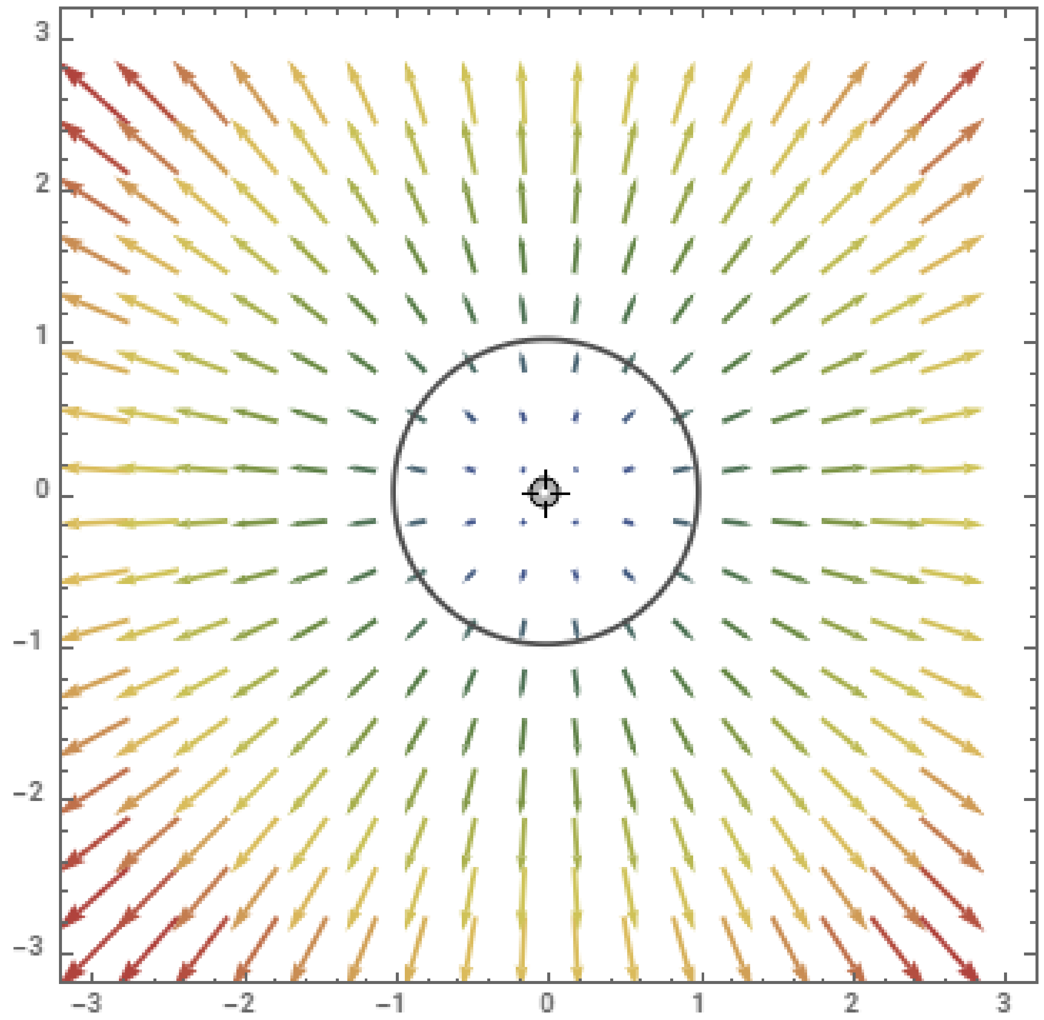
$$\theta = \begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix} \quad \Rightarrow \theta^t\,\theta = \theta_0^2 + \theta_1^2$$

If you imagine standing at a point (θ_0, θ_1),
$\nabla(\theta^T\theta)$ tells you which direction you should
travel to increase the value of $\theta^T\theta$ most rapidly.

$$\nabla(\theta^T\theta) = \begin{bmatrix}\dfrac{\partial}{\partial(\theta_0)}(\theta^T\theta)\\ \dfrac{\partial}{\partial(\theta_1)}(\theta^T\theta)\end{bmatrix} = \begin{bmatrix}2\theta_0\\ 2\theta_1\end{bmatrix} \approx \begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix}$$

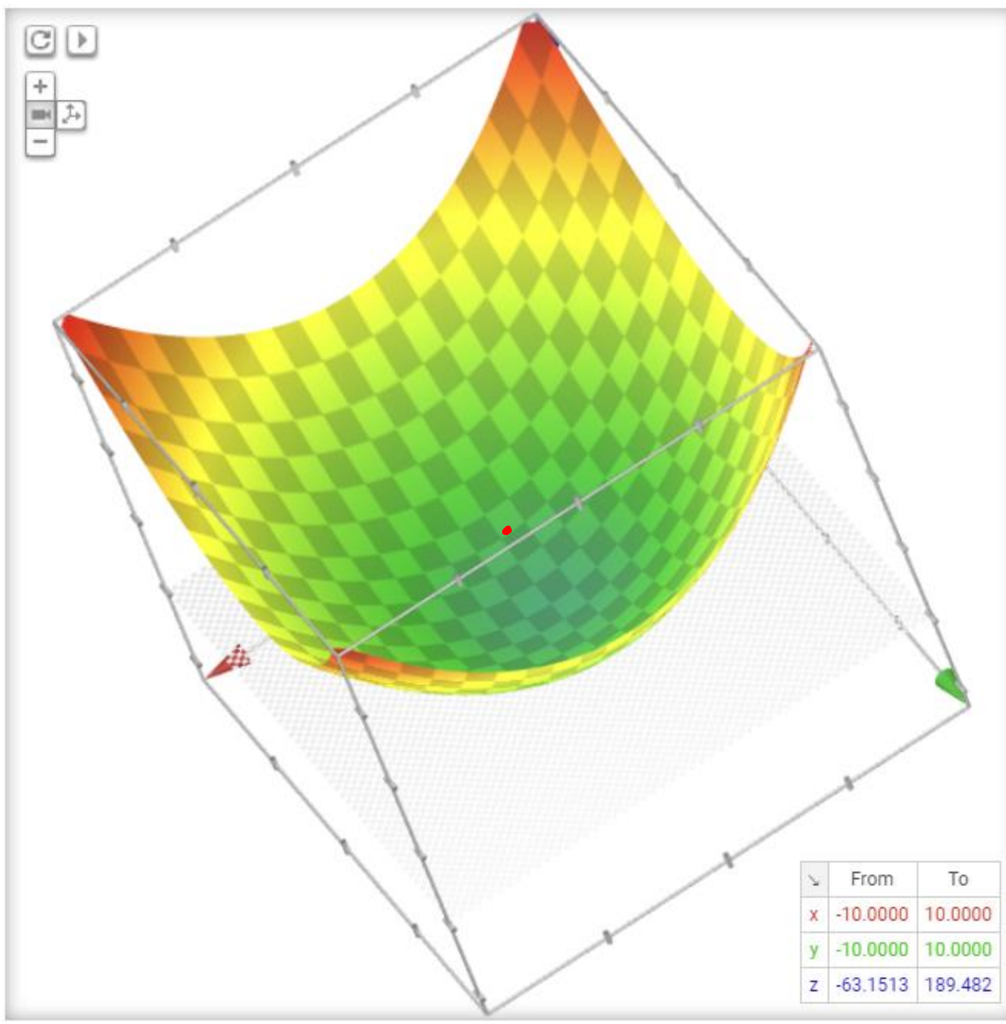
$\nabla(\theta^T\theta)$ is a vector field

any line passing through the center of the circle

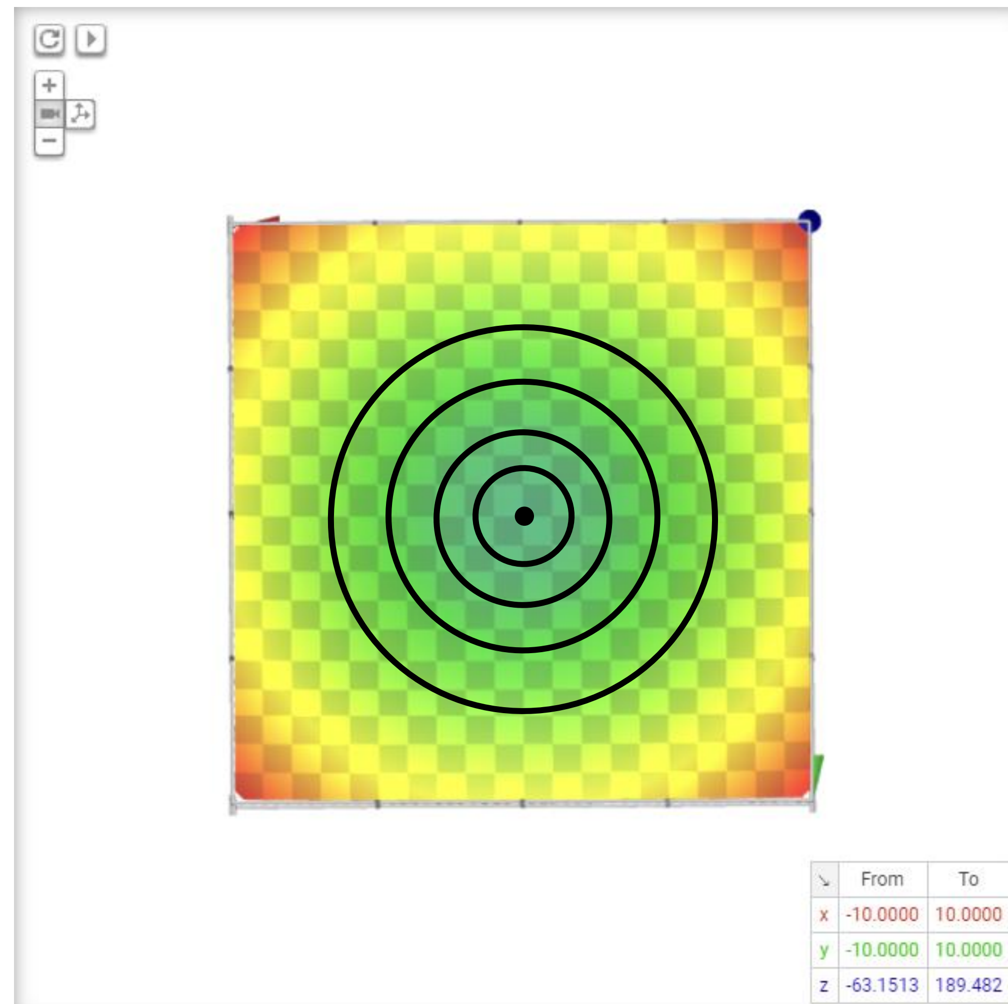


Plotting the regularization term $\theta^t\theta$

$\theta = \begin{bmatrix}\theta_0 \\ \theta_1\end{bmatrix}$ $\Rightarrow \theta^t\,\theta = \theta_0^2 + \theta_1^2 \longrightarrow$ eq. for a circle



3D view



Top view

Σ

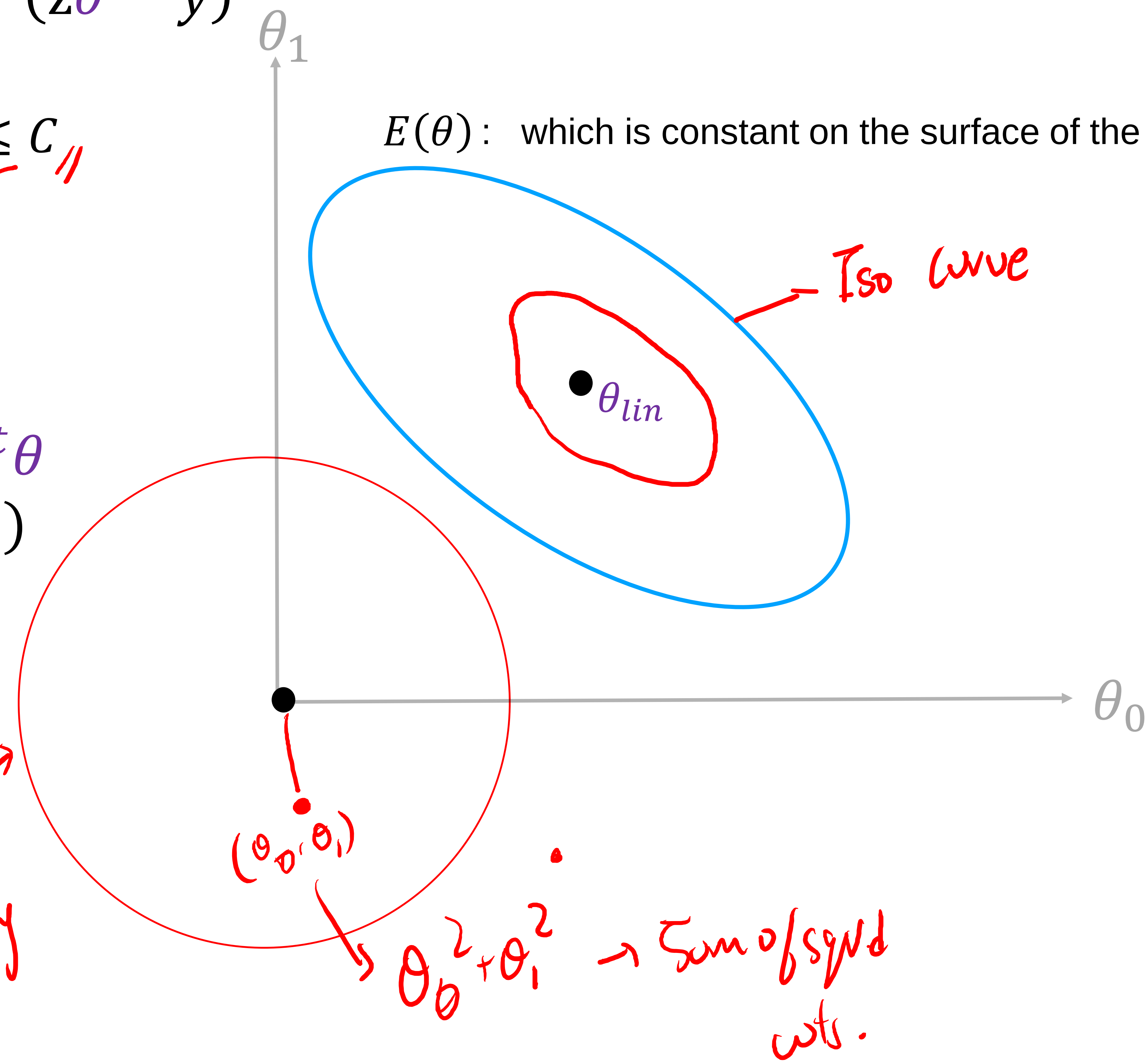
$$E(\theta) = \frac{1}{N}(z\theta - y)^T(z\theta - y)$$

θ_{lin} is the solution (min absolute)

Subject to $\theta^t\theta \le C$

$E(\theta)$: which is constant on the surface of the ellipsoid

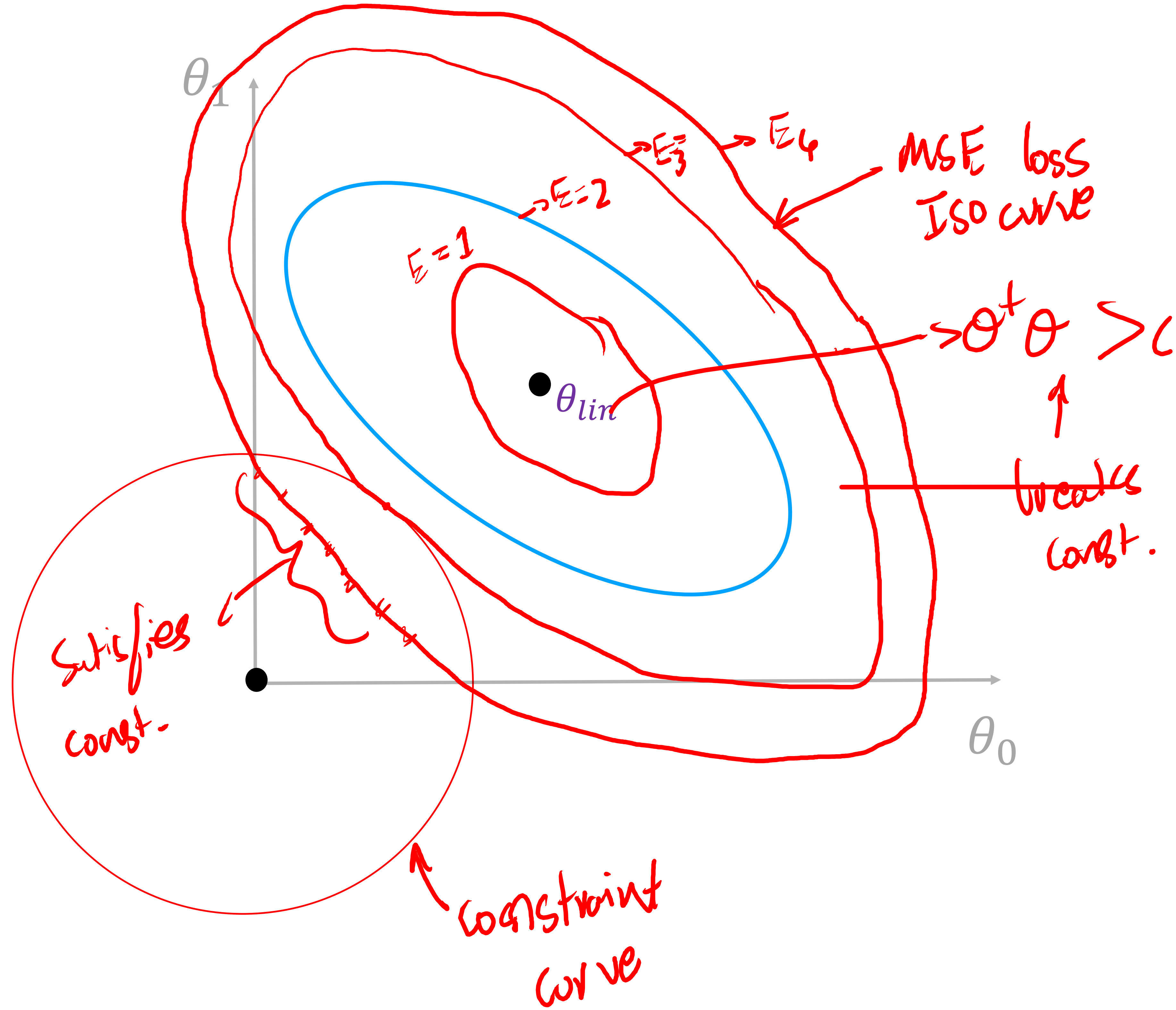
Find a solution in $\theta^t\theta$ that minimizes $E(\theta)$



Constraint boundary

Constraint and Loss

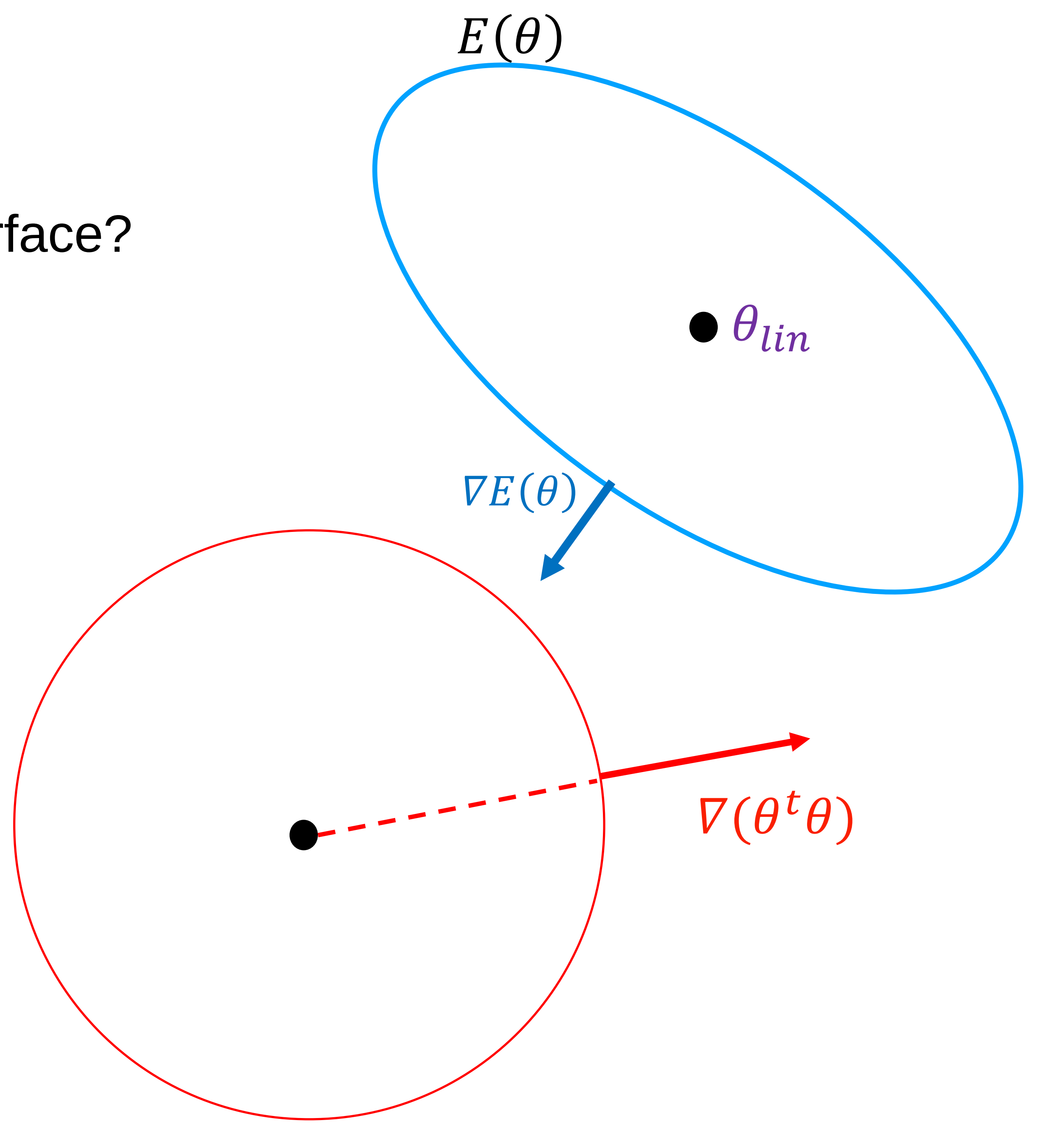
$\theta^t\theta \le c$



Considering the below $E(\theta)$ and C what is a θ candidate here?

∇E**:** the gradient (rate) in objective function which minimizes error (orthogonal to ellipse. Changes happen in orthogonal direction)

$$\theta^t\theta = Constraint = C$$



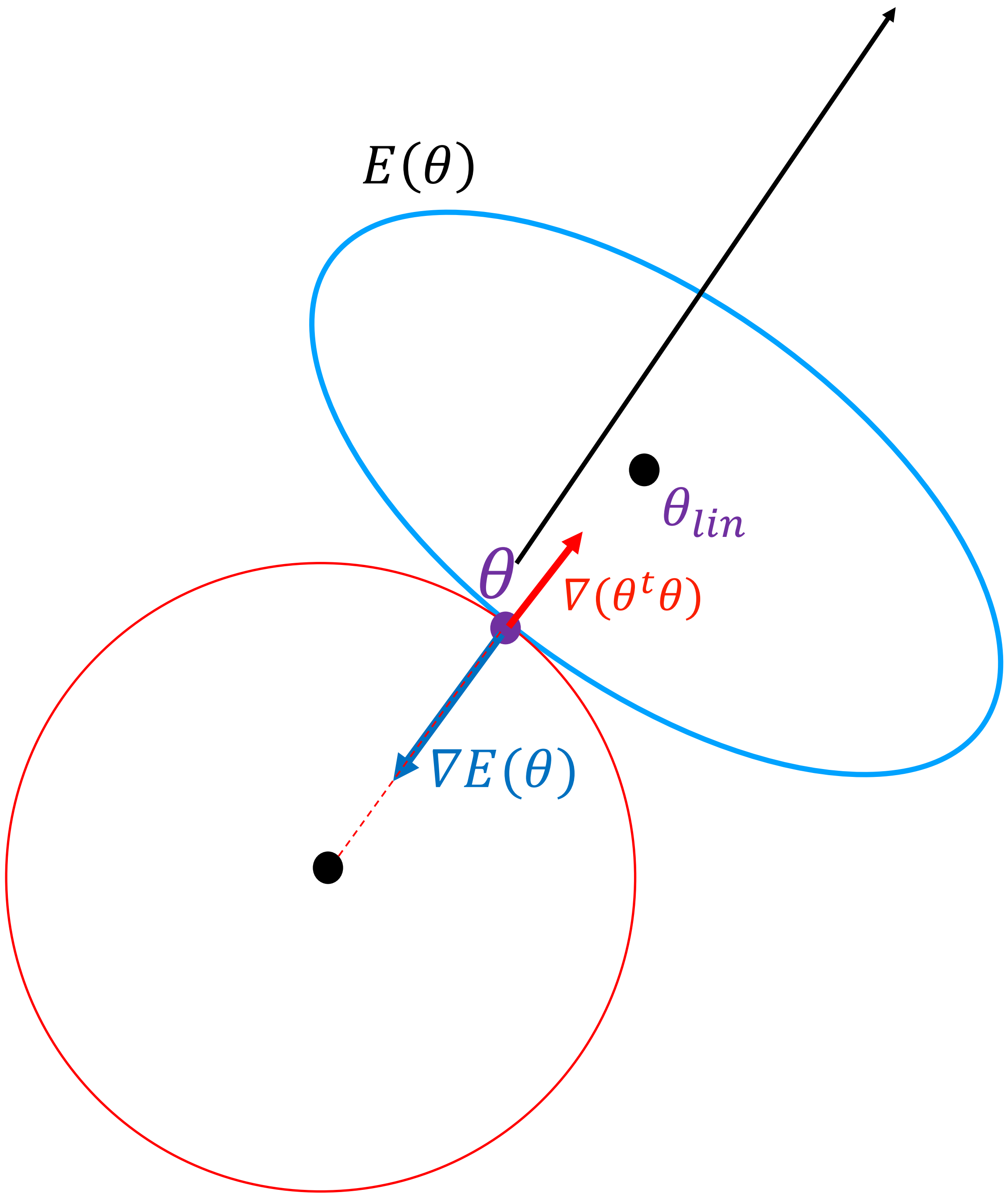
What is the orthogonal direction on the other surface?

It is just θ, a line passing through center of the circle

Applying a constrain $\theta^t\theta$, where the best solution happens?

On the boundary of the circle, as it is the closest one to the minimum absolute

Considering the below $E(\theta)$ and C what is the bew θ solution here?



$$\nabla E(\theta) \propto -\theta$$

$\nabla(\theta^t\theta) = \theta$

$$\nabla E(\theta) = -2\frac{\lambda}{N}\theta$$

$$\underbrace{\nabla E(\theta) + 2\frac{\lambda}{N}\theta = 0}$$

regularization term

Let's do integration

Minimize $E(\theta) + \frac{\lambda}{N}\theta^T\theta$

$C \uparrow \lambda \downarrow$

Outline

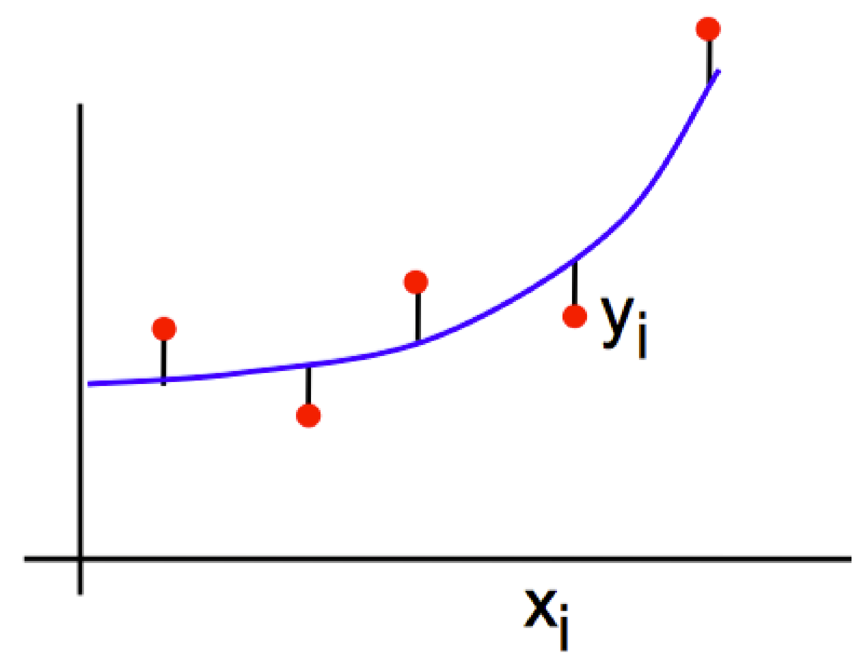
- Overfitting and regularized learning
- Ridge regression
- Lasso regression
- Determining regularization strength

Ridge Regression

- Cost function – squared loss:

target value

$$\widetilde{E}(\theta) = \underbrace{\frac{1}{N}\sum_{i=1}^{N}\{f(x_i,\theta) - y_i\}^2}_{\text{loss function}} + \underbrace{\frac{\lambda}{N}\|\theta\|^2}_{\text{regularization}}$$



- Regression function for x (1D):

$$f(x,\theta) = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + \cdots + \theta_d z_d + \epsilon = \boldsymbol{z\theta}$$

$z_n = x^n$

Solving for the Weights θ

Notation: write the target and regressed values as N-vectors

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_N \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} z(x_1)\theta \\ z(x_2)\theta \\ . \\ . \\ z(x_n)\theta \end{pmatrix} = z\theta = \begin{bmatrix} 1 & z_1(x_1) & \dots & z_d(x_1) \\ 1 & z_1(x_2) & \dots & z_d(x_2) \\ . & & & \\ . & & & \\ 1 & z_1(x_n) & \dots & z_d(x_n) \end{bmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ . \\ . \\ \theta_d \end{pmatrix}$$

d-dim

z is an $N \times$ D design matrix

e.g. for polynomial regression with basis functions up to x^2

$$z\theta = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ . & & . \\ . & & . \\ 1 & x_N & x_N^2 \end{bmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix}$$

data point

$$\tilde{E}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\{f(x_i, \overset{\hat{y}}{\theta}) - y_i\}^2 + \frac{\lambda}{N}\|\theta\|^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}(y_i - z_i\theta)^2 + \frac{\lambda}{N}\|\theta\|^2$$

$$= \frac{1}{N}(y_i - z\theta)^2 + \frac{\lambda}{N}\|\theta\|^2$$

matrix form

Now, compute where derivative w.r.t. θ is zero for minimum

$$\frac{\tilde{E}(\theta)}{d\theta} = -z^T(y - z\theta) + \lambda\theta$$

Hence

$$(z^T z + \lambda I)\theta = z^T y$$

no regurarization EQ.

$\theta = (Z^T Z)^{-1} Z^T Y$

$$\theta = (z^T z + \lambda I)^{-1} z^T y$$

D basis functions, N data points

$$\theta = (z^T z + \lambda I)^{-1} z^T \; y$$

$$[\;] = (\;\;) \quad (\;\;\;\;)[\;]$$

Dx1 DxD DxN Nx1

assume N>D

- This shows that there is a unique solution.

- If $\lambda = 0$ (no regularization), then

old unreg. sol.

$$\theta = (z^T z)^{-1} z^T y = z^+ y$$

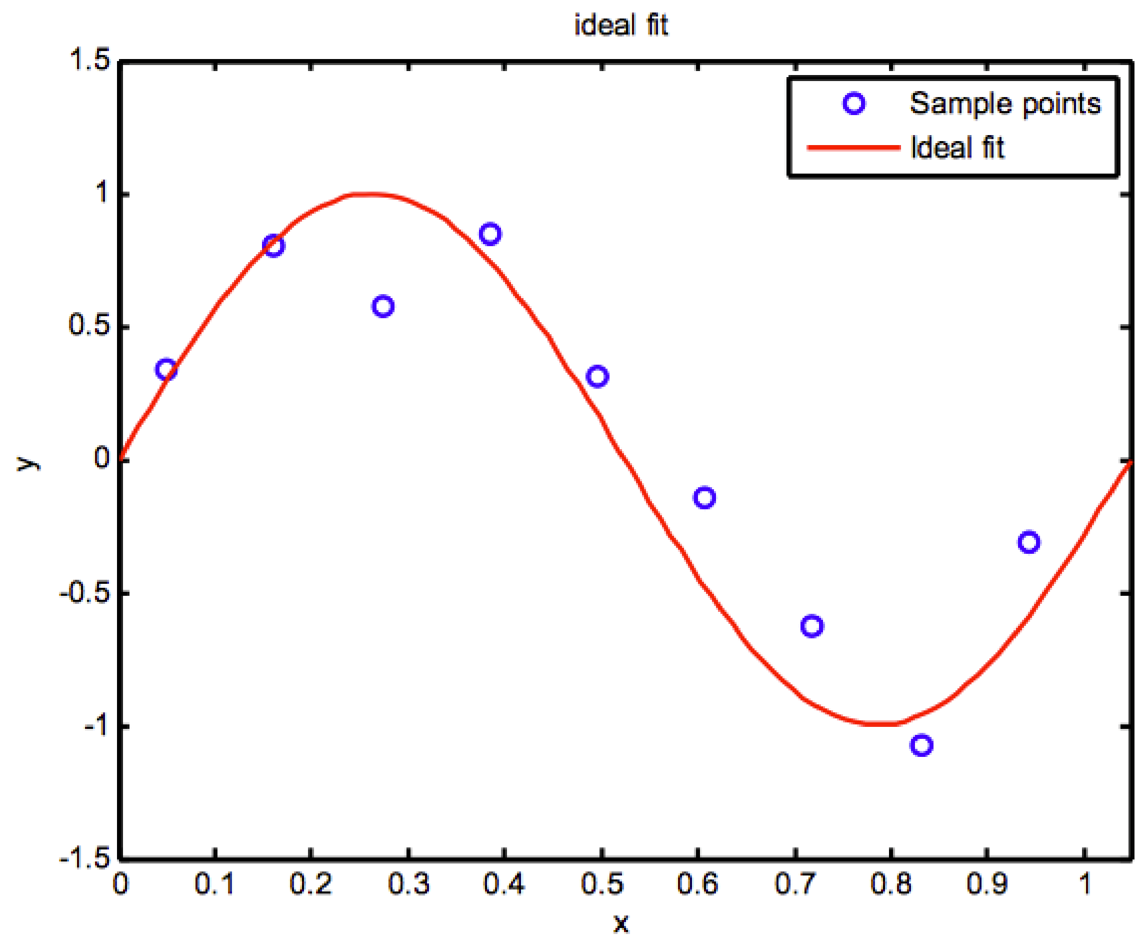
where z^+ is the pseudo-inverse of z (**pinv** in Matlab)

- Adding the term λI improves the conditioning of the inverse, since if z is not full rank, then $(z^T z + \lambda I)$ will be (for sufficiently large λ)

- As $\lambda \to \infty$, $\theta \to \frac{1}{\lambda} z^T y \to \mathbf{0}$

Ridge Regression Example

- The red curve is the true function (which is not a polynomial)
- The data points are samples from the curve with added noise in y.
- There is a choice in both the degree, D, of the basis functions used, and in the strength of the regularization



$$f(x,\theta) = z\theta \qquad z: x \to z \qquad \mathbb{R} \to \mathbb{R}^{D+1}$$

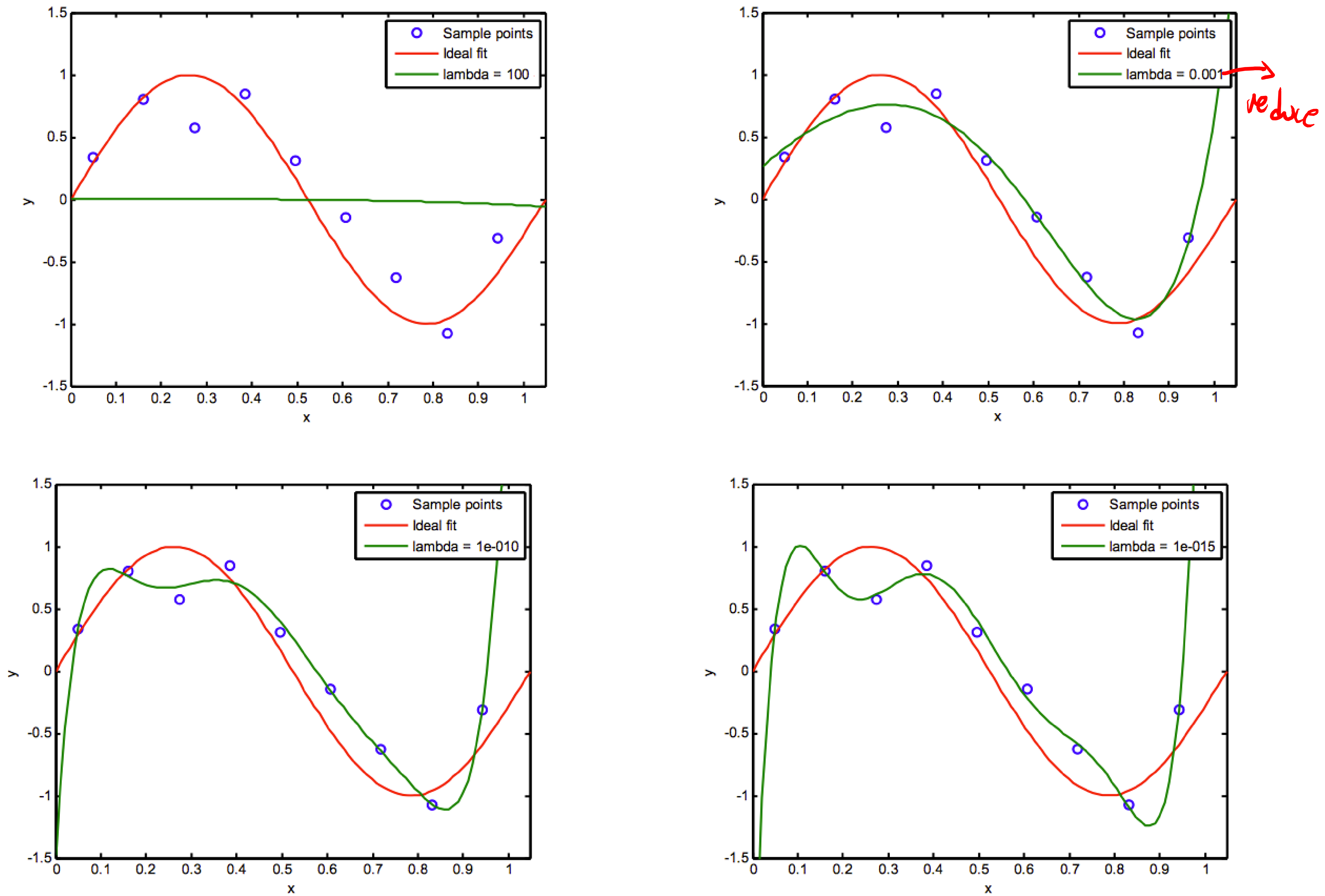
data points

$$\tilde{E}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\{f(x_i,\theta) - y_i\}^2 + \frac{\lambda}{N}\|\theta\|^2$$

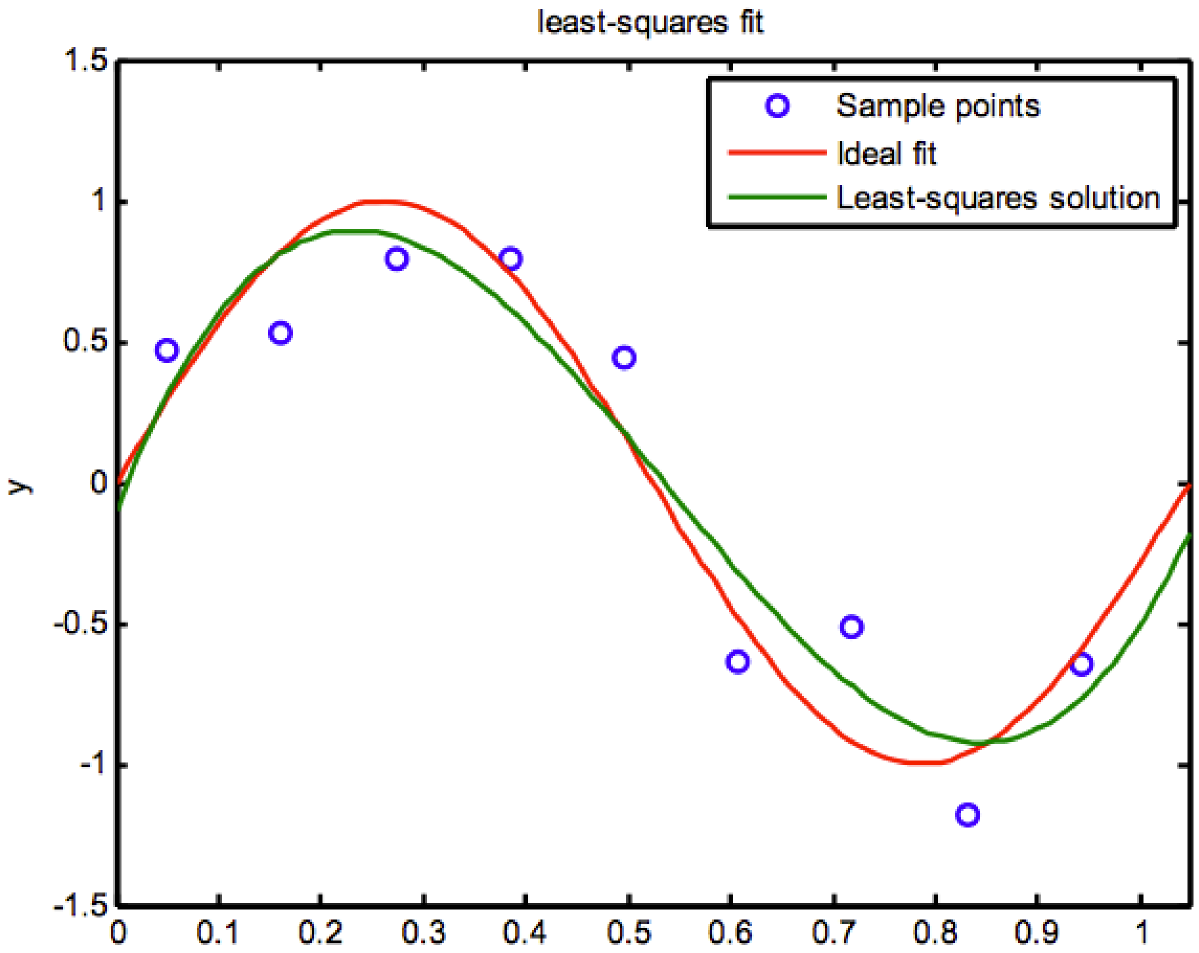
θ is a D+1 dimensional vector

$\hookrightarrow \sum_{D+1} \theta^2$

N = 9 samples, D = 7

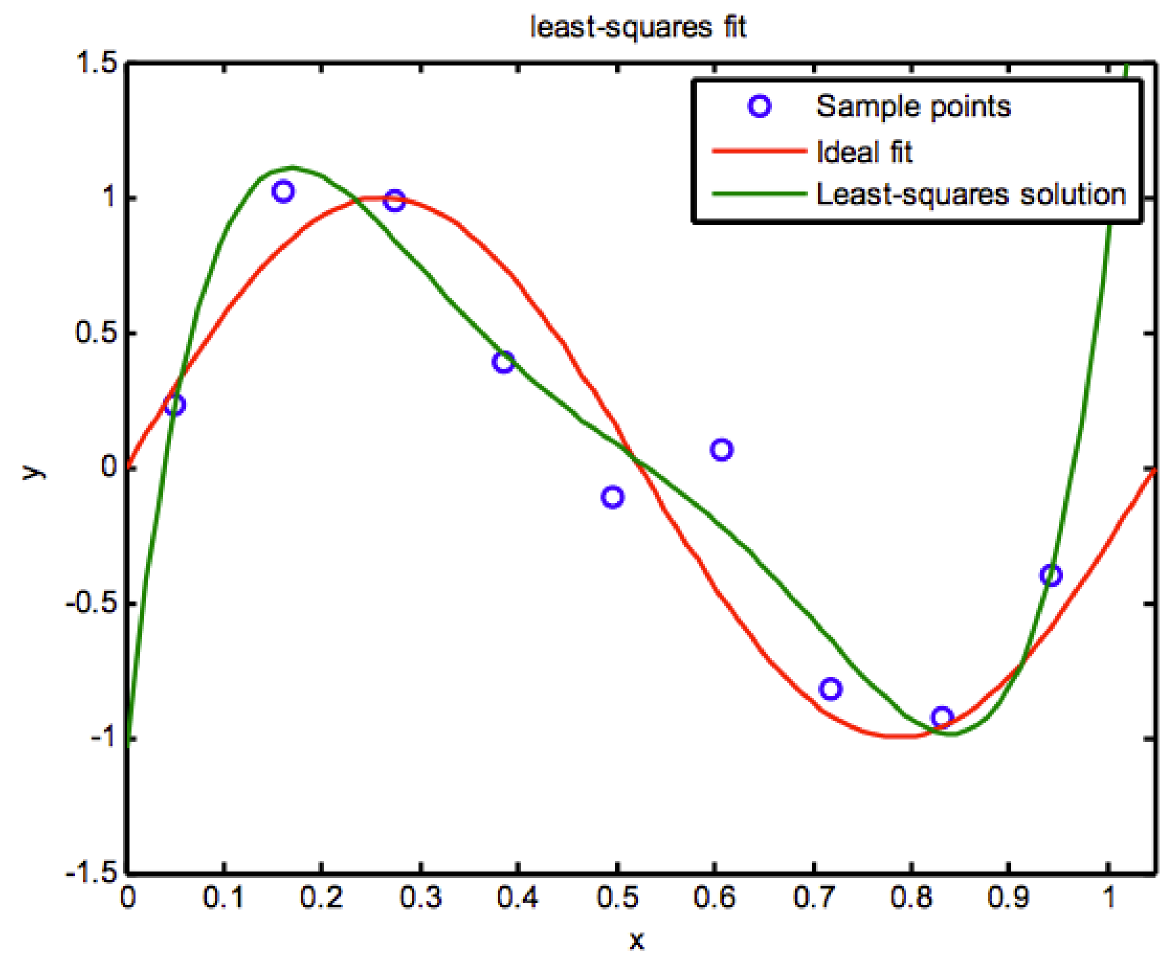


D = 3



least-squares fit
Sample points
Ideal fit
Least-squares solution
y
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

D = 5



least-squares fit
Sample points
Ideal fit
Least-squares solution
y
x
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Outline

- Overfitting and regularized learning
- Ridge regression
- Lasso regression
- Determining regularization strength

Regularized Regression

Minimize with respect to θ

MSE

regularization const./strength.

← loss by Regularized

$$\sum_{i=1}^{N} \underbrace{l\left(f(\mathbf{x}_i, \theta), y_i\right)}_{\text{loss function}} + \lambda \underbrace{R(\theta)}_{\text{regularization}}$$

- There is a choice of both loss functions and regularization
- So far we have seen – "ridge" regression
  - squared loss: $\sum_{i=1}^{N} (y_i - f(x_i, \theta))^2$
  - squared regularizer: $\lambda \|\theta\|^2$

Now let's look at another regularization choice.

The Lasso Regularization (norm one)

- LASSO = Least Absolute Shrinkage and Selection

Minimize with respect to θ

$$\sum_{i=1}^{N} \underbrace{l\left(f(\mathbf{x}_i, \theta), y_i\right)}_{\text{loss function}} + \lambda \underbrace{R(\theta)}_{\text{regularization}}$$

- This is a quadratic optimization problem
- There is a unique solution
- p-Norm definition: $\|\theta\|_p = \left(\sum_{j=1}^{d} |\theta_i|^p\right)^{\frac{1}{p}}$

Let's say we have two parameters (θ_0 and θ_1)

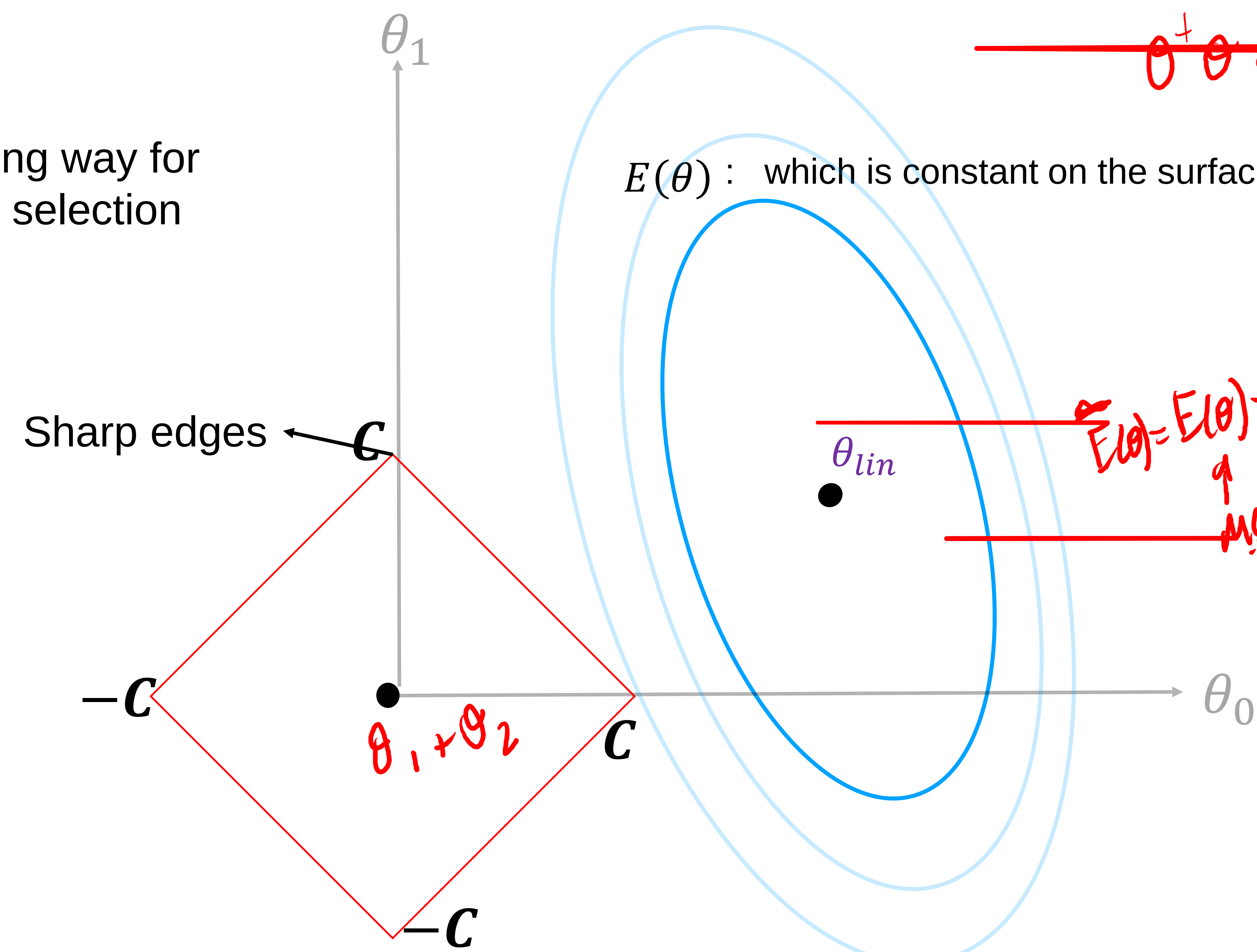
$$Minimize\ E(\theta) = \frac{1}{N}(\mathrm{z}w - y)^T(\mathrm{z}\theta - y)$$

$$\theta = \begin{bmatrix} \theta_0 \\ \mathbf{0} \end{bmatrix}$$

Subject to $\boldsymbol{\theta} \leq \boldsymbol{C}$

$\theta^T \theta \leq C \rightarrow$ ridge reg.

Interesting way for feature selection



$E(\theta) = E(\theta) + \lambda \|\theta\|$

MSE. Reg.

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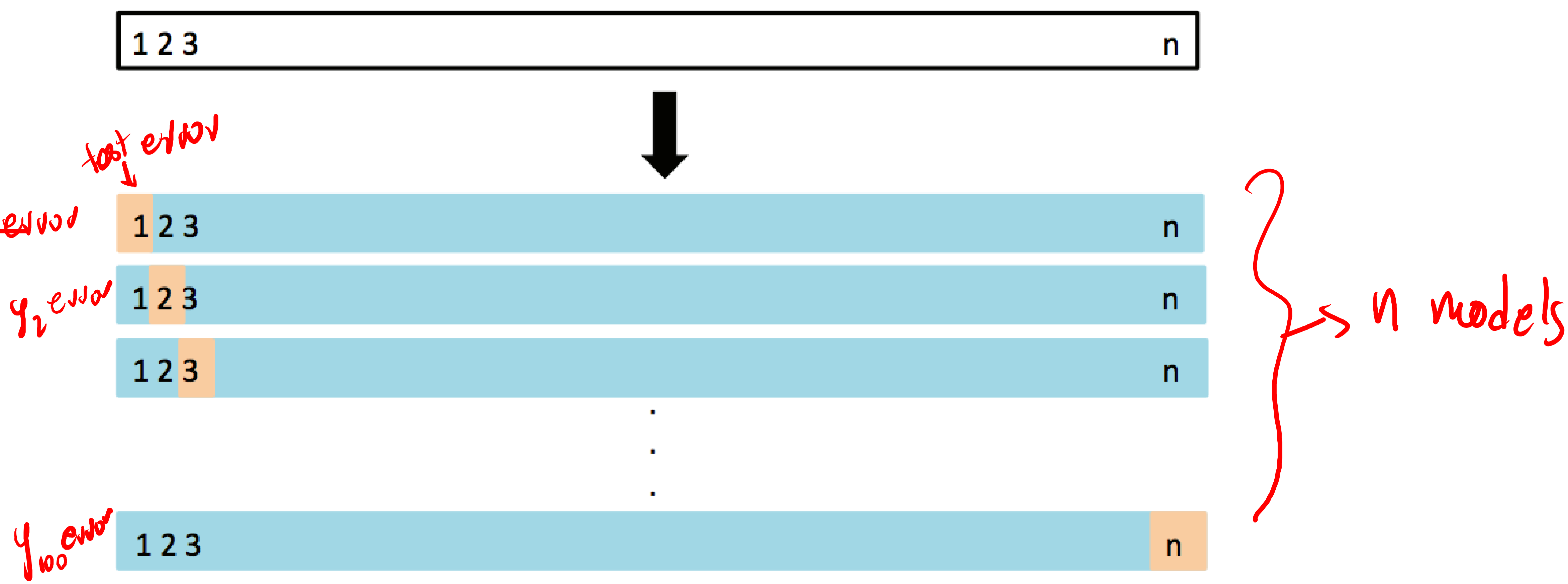
Leave-One-Out Cross Validation

For every $i = 1, \dots, n$:

- train the model on every point except i,
- compute the test error on the held out point.

Average the test errors. $\mathrm{CV}_{(n)} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i^{(-i)})^2$



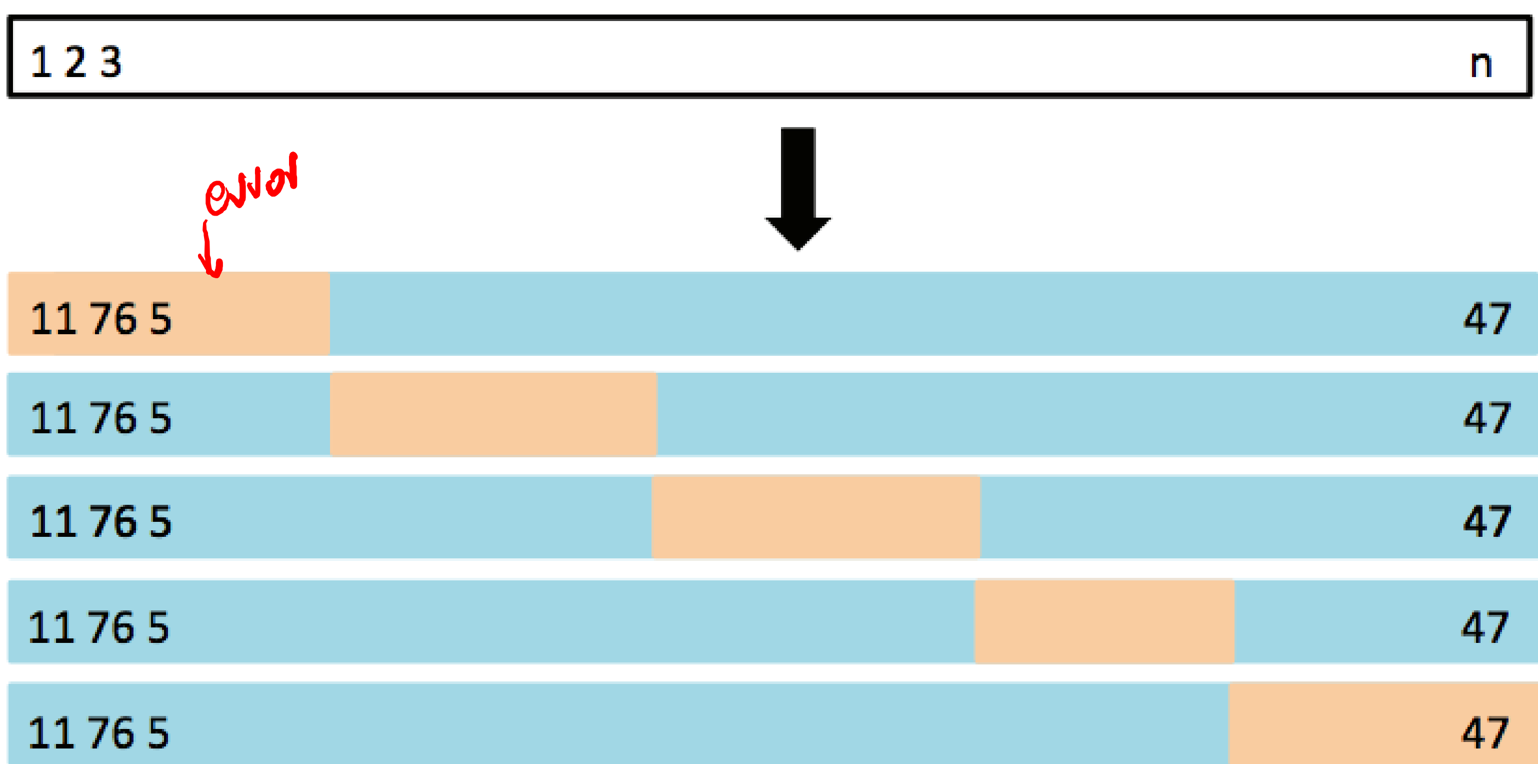
K-Fold Cross Validation

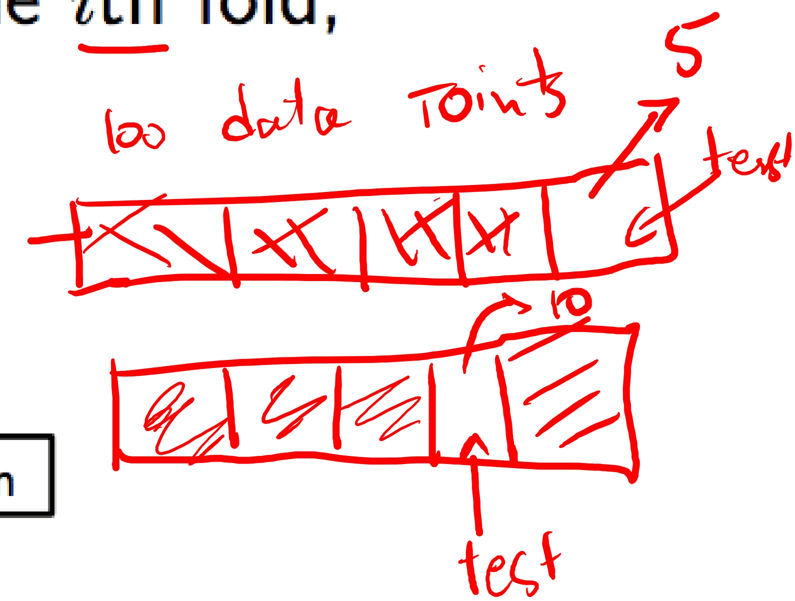
Split the data into k subsets or *folds*.

For every $i = 1, \ldots, k$:

- train the model on every fold except the ith fold,
- compute the test error on the ith fold.

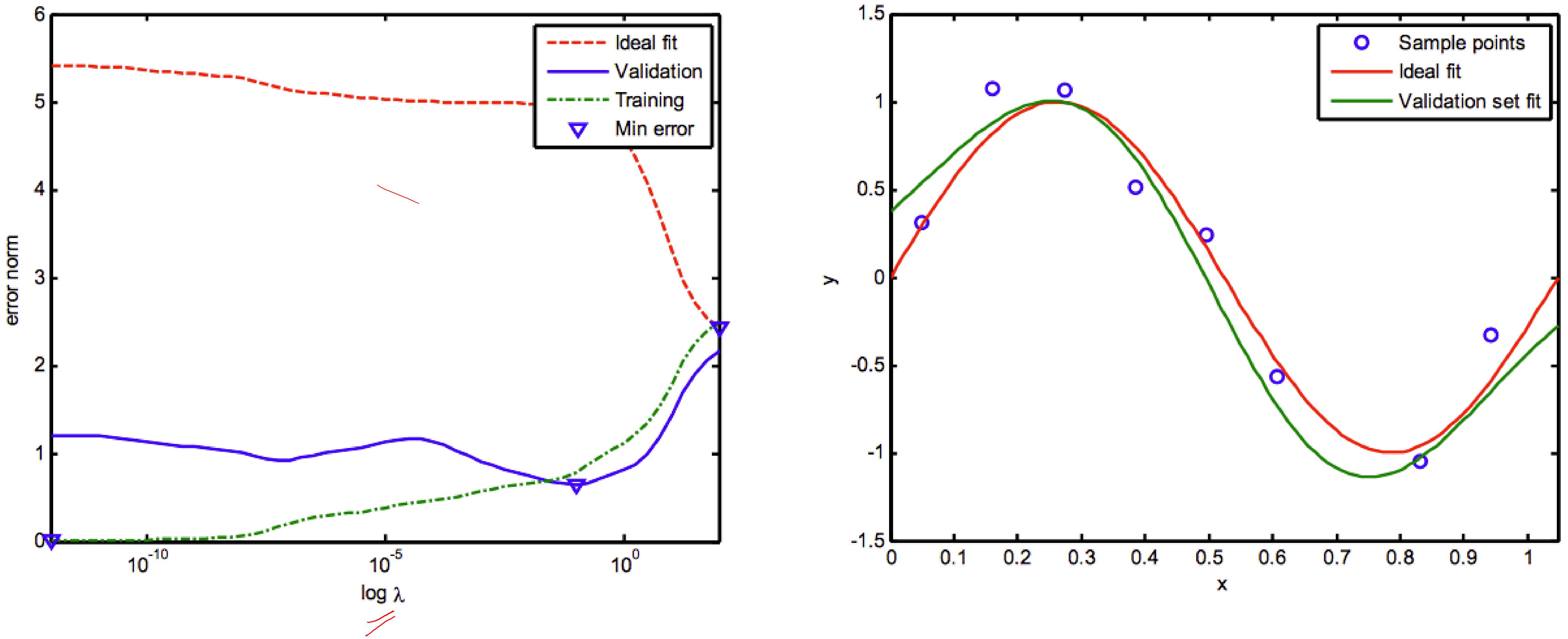
Average the test errors.





Cross Val error $= \sum_{i \in K} CV_i =$

Choosing λ Using Validation Dataset



Pick up the lambda with the lowest mean value of rmse calculated by Cross Validation approach

Take-Home Messages

- What is overfitting
- What is regularization
- How does Ridge regression work
- Sparsity properties of Lasso regression
- How to choose the regularization coefficient λ